October 1: Week 5 Problems

Problem 1*

Find which positive integers n have the property that there exist choices of the $\pm$ signs for which

$$\pm 1 \pm 2 \pm 3 \pm \cdots \pm (n-1) \pm n = 0$$

Problem 2*

Suppose that x is a real number such that $x + \frac{1}{x}$ is an integer. Prove that $x^n + \frac{1}{x^n}$ is an integer for every integer n.

Problem 3 (Putnam 2010)

Given that A, B, and C are noncollinear points in the plane with integer coordinates such that the distances AB, AC, and BC are integers, what is the smallest possible value of AB?

Problem 4 (Putnam 2013)

A regular icosahedron is a convex polyhedron having 12 vertices and 20 faces; the faces are congruent equilateral triangles. On each face of a regular icosahedron is written a nonnegative integer such that the sum of all 20 integers is 39. Show that there are two faces that share a vertex and have the same integer written on them.

Problems marked with an $*$ are from Ronnie Pavlov's Putnam Class at the University of Denver

Hints:

Problem 1

What is the sum of the numbers that have the same sign?

Problem 2

If you know $x^2 + \frac{1}{x^2}$ is also an integer, can you futher show $x^3 + \frac{1}{x^3}$ is an integer?

Problem 3

If $AB = 1$ or $AB = 2$, what does that make $|AC - BC|$?

Problem 4

Take each vertex and add all 5 numbers adjacent to it. Do this for each of the 12 vertices and add all 12 sums up together. What will this total sum be? Now, if each vertex had distinct numbers, what would the smallest possible total sum be?